A# Key

Induction Review Sheet

Prove the following using mathematical induction.

1) $1+4+7+\ldots+(3n-2)=\frac{n(3n-1)}{2}$ (Prove: $S_n = \frac{n(3n-1)}{2}$)

When $n=1$, $S_1 = a_1 = 1$ So this is true.

$$S_1 = \frac{1(3(1)-1)}{2} = \frac{1(3-1)}{2} = 1$$

Assume this is true when $n=K$. In other words, $S_K = \frac{K(3K-1)}{2}$. I want to show this is true when $n=K+1$.

$$S_{K+1} = S_K + a_{K+1}$$

$$= \frac{K(3K-1)}{2} + 3(K+1)-2$$

$$= \frac{3K^2-K}{2} + 3K+1$$

$$= \frac{3K^2-K+6K+2}{2}$$

$$= \frac{3K^2+5K+2}{2}$$

$$= \frac{(K+1)(3K+2)}{2}$$

$$= \frac{(K+1)(3(K+1)-1)}{2}$$

$\therefore 1+4+7+\cdots+(3n-2) = \frac{n(3n-1)}{2}$ for all n. ☺

2) $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\ldots+\frac{1}{4n^2-1}=\frac{n}{2n+1}$ (Prove $S_n = \frac{n}{2n+1}$)

When $n=1$, $S_1 = a_1 = \frac{1}{3}$

$S_1 = \frac{1}{2(1)+1} = \frac{1}{3}$ so this is true.

Assume this is true when $n=K$. In other words, $S_K = \frac{K}{2K+1}$. I want to show this is true when $n=K+1$.

$$S_{K+1} = S_K + a_{K+1}$$

$$= \frac{K}{2K+1} + \frac{1}{4(K+1)^2-1}$$

$$= \frac{K}{2K+1} + \frac{1}{4(K^2+2K+1)-1}$$

$$= \frac{K}{2K+1} + \frac{1}{4K^2+8K+3}$$

$$= \frac{K}{2K+1} + \frac{1}{(2K+1)(2K+3)}$$

$$= \frac{K(2K+3)}{(2K+1)(2K+3)} + \frac{1}{(2K+1)(2K+3)}$$

$$= \frac{2K^2+3K+1}{(2K+1)(2K+3)}$$

$$= \frac{\cancel{(2K+1)}(K+1)}{\cancel{(2K+1)}(2K+3)}$$

$$= \frac{K+1}{2(K+1)+1}$$

$\therefore \frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\cdots+\frac{1}{4n^2-1} = \frac{n}{2n+1}$ for all n. :)

3) Prove that $\sum_{a=1}^{n}(2a)^3 = 2n^2(n+1)^2$ ($\text{Prove } S_n = 2n^2(n+1)^2$)

When $n=1$, $S_1 = a_1 = (2\cdot 1)^3 = 8$
$S_1 = 2(1)^2(1+1)^2 = 2(1)(2)^2 = 8$ so this is true.

Assume this is true when $n=K$. In other words, $S_K = 2K^2(K+1)^2$. I want to show this is true when $n = K+1$.

$$S_{K+1} = S_K + a_K$$

$$= 2K^2(K+1)^2 + (2(K+1))^3$$

$$= (K+1)^2\left[2K^2 + 2^3(K+1)\right]$$ (Note: I factored out $(K+1)^2$)

$$= (K+1)^2(2K^2 + 8K + 8)$$

$$= 2(K+1)^2(K^2 + 4K + 4)$$

$$= 2(K+1)^2(K+2)^2$$

$$= 2(K+1)^2((K+1)+1)^2$$

$\therefore \sum_{a=1}^{n}(2a)^3 = 2n^2(n+1)^2$ for all n. :)

In problems 4-5, write out the first few terms of the sequence and suggest and prove a formula in terms of n for the nth term of a_n.

4) Given: $a_1 = 2$
$a_n = 3a_{n-1} + 2$

$a_2 = 3(2) + 2 = 6 + 2 = 8$

$a_3 = 3(8) + 2 = 24 + 2 = 26$

$a_4 = 3(26) + 2 = 78 + 2 = 80$

$\boxed{a_n = 3^n - 1}$

When $n = 1$, $a_1 = 2$ by def.

$a_1 = 3^1 - 1 = 3 - 1 = 2$ so this is true.

Assume this is true when $n = k$. In other words, $a_k = 3^k - 1$. I want to show this is true when $n = k+1$.

$$a_{k+1} = 3(a_{k+1-1}) + 2$$

$$= 3a_k + 2$$

$$= 3(3^k - 1) + 2$$

$$= 3 \cdot 3^k - 3 + 2$$

$$= 3^{k+1} - 1$$

$\therefore a_n = 3^n - 1$ for all n. :)

5) Given: $a_1 = \frac{1}{2}$

$a_n = \frac{n}{n+1}(a_{n-1}+1)$

$$a_2 = \frac{2}{3}\left(\frac{1}{2}+1\right) = \frac{2}{3}\left(\frac{3}{2}\right) = 1$$
$$a_3 = \frac{3}{4}(1+1) = \frac{3}{4}(2) = \frac{3}{2}$$
$$a_4 = \frac{4}{5}\left(\frac{3}{2}+1\right) = \frac{4}{5}\left(\frac{5}{2}\right) = 2$$
$$a_5 = \frac{5}{6}(2+1) = \frac{5}{6}(3) = \frac{5}{2}$$

$$\boxed{a_n = \frac{n}{2}}$$

When $n=1$, $a_1 = \frac{1}{2}$ by def.

$a_1 = \frac{1}{2}$ so this is true.

Assume this is true when $n=k$. In other words, $a_k = \frac{k}{2}$. I want to show this is true when $n=k+1$.

$$a_{k+1} = \frac{k+1}{k+1+1}\left(a_{k+1-1}+1\right)$$
$$= \frac{k+1}{k+2}\left(a_k+1\right)$$
$$= \frac{k+1}{k+2}\left(\frac{k}{2}+\frac{2}{2}\right)$$
$$= \frac{k+1}{\cancel{k+2}}\left(\frac{\cancel{k+2}}{2}\right)$$
$$= \frac{k+1}{2}$$

$\therefore a_n = \frac{n}{2}$ for all n. :)

In problems 6-7, write out the first few terms of the sequence and suggest and prove a formula in terms of n for the nth partial sum S_n.

6) $\sum_{a=1}^{n} 9 \bullet 10^{a-1} = 9 + 90 + 900 + \cdots + 9 \cdot 10^{n-1}$

$S_1 = 9$

$S_2 = 9 + 90 = 99$

$S_3 = 9 + 90 + 900 = 999$

$S_4 = 9999$

$\boxed{S_n = 10^n - 1}$

When $n=1$, $S_1 = a_1 = 9$ so this is true.

$S_1 = 10^1 - 1 = 9$

Assume this is true when $n = k$. In other words, $S_k = 10^k - 1$. I want to show this is true when $n = k+1$.

$$
\begin{aligned}
S_{k+1} &= S_k + a_{k+1} \\
&= 10^k - 1 + 9 \cdot 10^{k+1-1} \\
&= 10^k - 1 + 9 \cdot 10^k \\
&= 10 \cdot 10^k - 1 \\
&= 10^{k+1} - 1
\end{aligned}
$$

$\therefore \sum_{a=1}^{n} 9 \cdot 10^{a-1} = 10^n - 1$ for all n. ☺

7) Given: $a_1 = 1$
$a_n = 2n - 1$

$a_2 = 2(2) - 1 = 3$
$a_3 = 2(3) - 1 = 5$
$a_4 = 2(4) - 1 = 7$

$S_1 = 1$
$S_2 = 1 + 3 = 4$
$S_3 = 1 + 3 + 5 = 9$
$S_4 = 1 + 3 + 5 + 7 = 16$

$\boxed{S_n = n^2}$

When $n = 1$, $S_1 = a_1 = 1$
$S_1 = 1^2 = 1$ so this is true.

Assume this is true when $n = k$. In other words, $S_k = k^2$. I want to show this is true when $n = k+1$.

$$S_{k+1} = S_k + a_{k+1}$$
$$= k^2 + 2(k+1) - 1$$
$$= k^2 + 2k + 1$$
$$= (k+1)^2$$

$\therefore S_n = n^2$ for all n. :)